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A note on disturbances in slightly supercritical plane Poiseuille flow

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The evolution of disturbances after a laminar, slightly supercritical flow between parallel planes is disturbed is considered as an initial-value problem. An asymptotic solution of the disturbances for large time possesses the same characteristic features as the turbulent spots observed by Emmons (1951).

Introduction

Attempts have been made in the past to describe the development of three-dimensional disturbances in laminar parallel flow. Benjamin (1961) and Criminale & Kovasznay (1962) had investigated this question in some detail and reported some important conclusions. Here, the problem is treated by the initial-value approach following the ideas of Case (1960, 1962) and Dikii (1960). It is found that, for large time after a laminar, slightly supercritical plane Poiseuille flow is disturbed, the disturbance is confined to a triangular region resembling the turbulent spots observed by Emmons (1951).

Analysis

Consider laminar flow between two parallel planes. Let $\overline{u}(z)$ be the undisturbed velocity and choose a co-ordinate system such that x is in the direction of flow, z is in the direction normal to the planes, and y is perpendicular to x and z. By eliminating pressure from the dimensionless, linearized Navier–Stokes equations, the velocity component w in the z-direction is governed by the single equation

$$\frac{\partial}{\partial t}(\nabla^2 w) + \overline{u}\frac{\partial}{\partial x}(\nabla^2 w) - \overline{u}''\frac{\partial w}{\partial x} = \frac{1}{R}\nabla^4 w. \tag{1}$$

The boundary conditions on w are

$$w = \partial w/\partial z = 0 \quad (z = \pm 1). \tag{2}$$

The initial-value problem of (1) and (2) can be solved by the Fourier–Laplace transform technique, which reduces essentially to the construction of a Green's function $G(z, z_0; \alpha, \beta, p)$ satisfying the following properties:

$$(D^2-\alpha^2-\beta^2)^2 G - i\alpha R\left[\left(\overline{u}+\frac{p}{i\alpha}\right)(D^2-\alpha^2-\beta^2)\,G - \overline{u}''G\right] = \delta(z-z_0), \tag{3}$$

$$G = DG = 0, \quad z = \pm 1, \quad D = d/dz,$$

where α, β and p are the transformed variables of x, y and t.

If ϕ_i ($i = 1$ to 4) are four linearly independent solutions of the homogeneous equation (3), a linear combination of them can be used to construct the function G in the usual way (Case 1960). The following properties of G can be deduced.

(i) The poles of G in the p-plane are the eigenvalues of the normal mode method.

(ii) If $p_0 = \gamma(\alpha,\beta) - i\omega(\alpha,\beta)$ is a pole, $\tilde{p}_0 = \gamma(-\alpha,\beta) + i\omega(-\alpha,\beta)$ is also a pole.

(iii) When evaluated at a pole, the function $(p-p_0)G$ has the functional form $\phi_e(z,\alpha,\beta,p_0)\,\Delta(z_0,\alpha,\beta,p_0)$, where ϕ_e is the eigenfunction of (3).

The general solution of (1) and (2) is

$$w = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{+1}\int_{\Gamma-i\infty}^{\Gamma+i\infty} G(z,z_0;\alpha,\beta,p)\,\overline{\phi}(\alpha,\beta,z_0)\exp\left(i\alpha x + i\beta y + pt\right) dp\, dz_0\, d\alpha\, d\beta, \tag{4}$$

where $\overline{\phi}(\alpha,\beta,z_0)$ depends on the initial conditions.

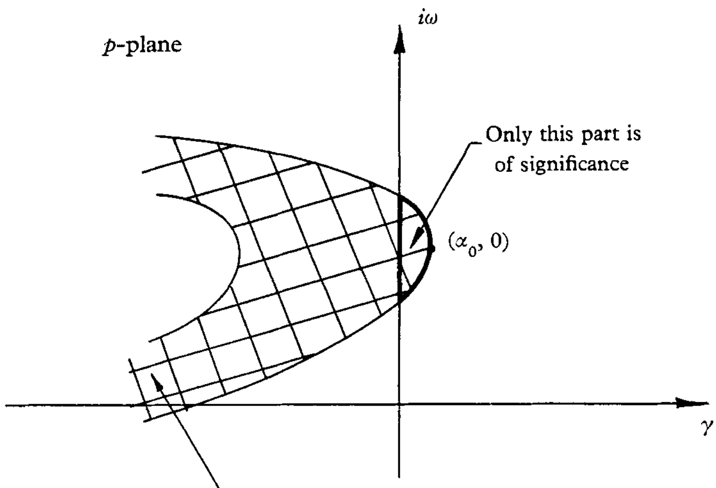

FIGURE 1. p-plane.

Consideration will now be restricted to slightly supercritical flow when only a pair of simple poles of G lie on the right half of the p-plane for a certain range of α,β. The study of Squire (1933) and Watson (1960) on the eigenvalue problem associated with (3) shows that at slightly supercritical conditions $\gamma(\alpha,\beta)$ has a positive maximum at $\alpha = \alpha_0, \beta = 0$, where α_0 is the wave-number of maximum amplification for two-dimensional disturbance. Therefore, for large t, only values of α and β near this maximum point make significant contributions to the integral (figure 1). On deforming the contour integral of (3) to the left in the p-plane for large t:

$$w(x,y,z,t) \sim I_1 + I_2 \quad \text{(from the pair of poles)}, \tag{5}$$

$$I_1 \sim A\phi_e(z,\alpha_0,0,p_0(\alpha_0,0)) \iint_{-\infty}^{\infty} \exp\{i\alpha x + i\beta y + \gamma(\alpha,\beta)t - i\omega(\alpha,\beta)t\}\, d\alpha\, d\beta;$$

$A(\alpha_0, 0)$ is a constant. (6)

In (6), γ and ω can be approximated by

$$\gamma(\alpha,\beta) = \gamma(\alpha_0,0) - \frac{1}{2}\left|\left(\frac{\partial^2\gamma}{\partial\alpha^2}\right)_0\right|(\alpha-\alpha_0)^2 - \frac{1}{2}\left|\left(\frac{\partial^2\gamma}{\partial\beta^2}\right)_0\right|\beta^2 \tag{7}$$

and

$$\omega(\alpha,\beta) = \omega(\alpha_0,0) + \left(\frac{\partial\omega}{\partial\alpha}\right)_0(\alpha-\alpha_0).$$

(Note: in (7),

$$\left(\frac{\partial\gamma}{\partial\beta}\right)_0 = \left(\frac{\partial^2\gamma}{\partial\alpha\,\partial\beta}\right)_0 = \left(\frac{\partial\omega}{\partial\beta}\right)_0 = 0$$

since β appears as β^2 in (3).)

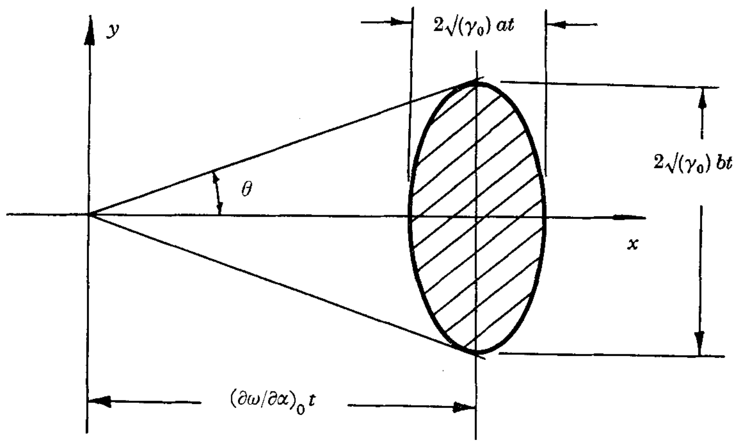

FIGURE 2. Propagation of disturbance at slightly supercritical flow.

Substitute (7) into (6) and integrate, and note that I_2 yields the complex conjugate of I_1:

$$w \sim \mathrm{Re}\left\{\frac{8\pi A\phi_e(z)\exp\{i(\alpha_0 x-\omega_0 t)\}}{abt}\right\}\exp\left\{\left[\gamma_0 - \frac{1}{a^2}\left(\frac{x}{t}-\left(\frac{\partial\omega}{\partial\alpha}\right)_0\right)^2 - \frac{1}{b^2}\left(\frac{y}{t}\right)^2\right]t\right\},$$

$$a^2 = 2\left|\left(\frac{\partial^2\gamma}{\partial\alpha^2}\right)_0\right|,\quad b^2 = 2\left|\left(\frac{\partial^2\gamma}{\partial\beta^2}\right)_0\right|. \tag{8}$$

For large t, the value of w as given by (8) is exponentially small if (x,y) is outside the region defined by

$$\frac{1}{a^2}\left(\frac{x}{t}-\left(\frac{\partial\omega}{\partial\alpha}\right)_0\right)^2 + \frac{1}{b^2}\left(\frac{y}{t}\right)^2 = \gamma_0; \tag{9}$$

(8) and (9) show that the disturbance is confined to an ellipse with semi-axes $(\gamma_0)^{\frac{1}{2}}at$ and $(\gamma_0)^{\frac{1}{2}}bt$ in the x and y directions. This ellipse subtends an angle 2θ at the origin (figure 2), where

$$\tan\theta = \left\{\frac{2\gamma_0\left|\left(\frac{\partial^2\gamma}{\partial\beta^2}\right)_0\right|}{\left(\frac{\partial\omega}{\partial\alpha}\right)_0^2 - 2\gamma_0\left|\left(\frac{\partial^2\gamma}{\partial\alpha^2}\right)_0\right|}\right\}^{\frac{1}{2}}. \tag{10}$$

The disturbance propagates downstream with velocity equal to $(\partial\omega/\partial\alpha)_0$. This is the group velocity of the most amplified two-dimensional wave.

To find the other velocity components u and v, the same Fourier–Laplace transform method can be used. The problem then reduces to the construction of the following Green's function $H(z, z_0; \alpha, \beta, p)$ satisfying

$$(D^2-\alpha^2-\beta^2)H - R(p+i\alpha_0\bar{u})H = \delta(z-z_0), \quad H = 0, \quad z = \pm 1. \tag{11}$$

The study of Squire (1933) shows that the poles of H in the complex p-plane always lie on the left half plane for all values of α, β. Therefore, at slightly supercritical flow for large t, u and v have the same general behaviour as w.

Concluding remarks

The above results show that, if a laminar flow at slightly supercritical condition is disturbed locally, then after a sufficiently long time (not too long to make the non-linear terms so far neglected important), a spot of disturbance will form. The spot is essentially an expanding ellipse which subtends an angle 2θ (equation 10) at the origin and sweeps out a triangular region in the (x, y)-plane in the course of time. The spot moves downstream with constant velocity $(\partial\omega/\partial\alpha)_0$ which is the group velocity of the most amplified two-dimensional wave as obtained by the normal mode method. It is interesting to point out that the characteristic features of the disturbance are the same as those of the turbulent spots observed by Emmons (1951). Although the present investigation is based on a linear analysis, yet with some reservation it is possible to use the above results to estimate the characteristic constants of Emmons's theory of turbulent spots theoretically.

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